Simulation of bipartite qudit correlations

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We present a protocol to simulate the quantum correlations of an arbitrary bipartite state, when the parties perform a measurement according to two traceless binary observables. We show that $\log(d)$ bits of classical communication is enough on average, where d is the dimension of both systems. To obtain this result, we use the sampling approach for simulating the quantum correlations. We discuss how to use this method in the case of qudits.

I. INTRODUCTION

In 1964 John Bell showed that the correlations exhibited by the EPR gendanken experiment [1, 2] could not be reproduced by a so-called local hidden variable model, that is, a model where the parties share an infinite amount of locally created hidden variables. This nonlocal aspect is one of the strangest properties of quantum physics, and understanding this notion remains an important problem. Recently, quantum information processing has provided a new point of view to understand quantum nonlocality. In particular, the framework of communication complexity has provided tools to study nonlocality. For example, two parties, whom we call Alice and Bob, cannot reproduce quantum correlations if they share only hidden random variables (shared randomness), but in some cases, if they are allowed to use some additional resources, it becomes possible for them to reproduce the quantum correlations. It is precisely this amount of additional resources which we consider here; they allow us to quantify quantum nonlocality.

The problem of reproducing the statistics of projective measurements on the singlet has been widely studied, with communication as the additional resource. In 1992, Maudlin [3] presented a protocol in the case of mesurements in the real plane and proved an average-case communication upper bound of 1.17 bits, and independently in 1999, Brassard, Cleve and Tapp gave a protocol, together with a worst-case communication upper bound of 8 bits, for arbitrary projective measurements [4]. In 2000, Steiner, independently of Maudlin, gave a protocol for projective measurements in the real plane with an average-case upper bound on communication of 1.48 bits, and Cerf, Gisin and Massar [5] proved that for an arbitrary projective measurements, 1.19 bits of communication sufficed on average. Recently, in 2003, Toner and Bacon have shown that one bit of communication is always enough to reproduce the quantum correlations for arbitrary projective measurements on the singlet state [6].

Some other resources have been used to simulate quantum correlations resulting from projective measurements on the singlet state. These include post-selection [7], and nonlocal boxes [8]. In 2005, we have shown that simulating these quantum correlations could be reduced to a sampling problem, from which we derived many of the above-mentioned protocols, in a unified framework [9].

Nevertheless, these results address the simplest scenario, that is, simulating the correlations resulting from measurements on the singlet state (mostly for projective measurements, with a few extensions to POVMs). There are few results about non-maximally entangled pairs, multiparty states, higher dimensional states, or more general measurements. One significant result in this direction is a protocol from Massar *et al* able to reproduce the correlations of arbitrary measurements on any entangled pair of d-dimensional states (qudits) using $O(d \log d)$ bits of communication but no local hidden variables [10].

In this paper, we use the sampling approach developed in [9], and generalize it to the case of a bipartite pair of arbitrary-dimension states (qudits). We study the case where the parties make a restricted type of measurement with only two opposite outcomes $\{1, -1\}$, that we call traceless binary observable, or TBO.

Furthermore we impose no constraint on the bipartite (pure) state whose correlations the parties wish to simulate; it could be maximally entangled or non-maximally entangled. For an arbitrary bipartite qudit pair, we show that $\log(d)$ bits of communication on average are enough to simulate the joint correlations of the outcomes (where the joint correlation is defined as the expectation value of the product of Alice's and Bob's outcome). In the special case of maximally entangled qudit pairs, our protocol also reproduces the marginal probabilities, and therefore the full probability distribution.

We will begin by describing the quantum correlations in arbitrary dimensions that we want to simulate classically. Then, using the sampling approach, we will present a generalization of the local biased hidden variable model for arbitrary dimensions, and present a classical protocol which uses $\log(d)$ bits of communication to simulate the joint quantum correlations of an arbitrary bipartite qudit pair.

II. THE QUANTUM CORRELATIONS

In this section we describe the system that we want to simulate classically using some communication. Two parties, Alice and Bob, share an arbitrary bipartite qudit pair. They each perform a measurement on their part, where the measurements are restricted to what we call traceless binary observables, described below.

We describe the measurements using observables instead of measurement operators. We restrict the measurements to be such that only two outcomes are possible, and these outcomes are equally likely when the measurement is applied to a maximally mixed state.

Definition 1 (Traceless Binary Observable). *An observable $\hat{A}$ is called a Traceless Binary Observable (TBO) if*

- *the observable is traceless (i.e. $Tr(\hat{A}) = 0$)*
- *the outputs of the measurement are two opposite values, more precisely $\hat{A}^2 = \mathbb{1}$.*

We describe the bipartite quantum correlations on qudits obtained when a TBO is applied to each part of the state.

Definition 2 (Qudits TBO experiment). *Two parties, Alice and Bob, share an arbitrary bipartite qudit pair, $|\psi\rangle \in \mathcal{H}^d \otimes \mathcal{H}^d$. Alice and Bob measure their part of the state according to their input, describing a TBO pair $\hat{A}, \hat{B}$. They then obtain measurement outcomes $A \in \{1, -1\}$ and $B \in \{1, -1\}$ respectively.*

We will use the following notation throughout the paper. Let $\mathcal{O}_d$ denote the space of TBOs over $\mathcal{H}^d$. We use $\mathbb{S}_n$ to denote the unit hypersphere in $\mathbb{R}^{n+1}$. (For example, $\mathbb{S}_2$ is the unit sphere in $\mathbb{R}^3$.) We will also use $\mathcal{S}_n$ to denote the surface area of $\mathbb{S}_n$.

Tsirelson showed that there is a function that maps TBOs over the Hilbert space (matrix operators) to points on the surface of a hypersphere, as follows [11]. This formulation of Tsirelson's theorem was pointed out to us by B. Toner [12] and appears in this form in [13].

Theorem 3 (Tsirelson). *For any $d > 0$, and $|\psi\rangle \in \mathcal{H}^d \otimes \mathcal{H}^d$, there is a function $\nu : \mathcal{O}_d \longrightarrow \mathbb{S}_{2d^2-1}$ such that the following holds. If $\hat{A}$ and $\hat{B}$ is a TBO pair over $\mathcal{H}^d$, and A, B are the outcomes of measuring $|\psi\rangle$ according to $\hat{A}$ and $\hat{B}$, then*

$$E(AB|\hat{A} \otimes \hat{B}) = \nu(\hat{A}) \cdot \nu(\hat{B}).$$

In the remainder of the paper, we use this theorem implicitly, and use the notation $\vec{a} = \nu(\hat{A})$, and $\vec{b} = \nu(\hat{B})$, to denote Alice's and Bob's inputs (or measurement), respectively. Furthermore, we call $E(AB|\hat{A} \otimes \hat{B})$ the joint quantum correlations.

III. THE CLASSICAL PROTOCOL

We present a protocol to simulate bipartite qudit joint quantum correlations as defined in the previous section, where the state $|\psi\rangle$ is an arbitrary state in $\mathcal{H}^d \otimes \mathcal{H}^d$. First, we generalize the sampling method introduced in [9].

A. Local hidden biased variable model

As in [9], we consider a model where Alice and Bob share random variables that can depend on Alice's and/or Bob's input, which we call a local biased random variable model.

We generalize the sampling theorem in [9] to arbitrary bipartite qudit states, with TBO measurements. We start with a technical lemma to compute the normalization factor of the biased distribution.

Lemma 4. $\int_{\mathbb{S}_n} \left| \vec{b} \cdot \vec{\lambda}_s \right| \, d\vec{\lambda}_s = \frac{2}{n}\mathcal{S}_{n-1}$.

Proof. Since $d\vec{\lambda_s} = d\theta_n\ sin(\theta_{n-1})d\theta_{n-1}\ sin^2(\theta_{n-2})d\theta_{n-2}...sin^{n-1}(\theta_2)d\theta_2\ sin^{n-1}(\theta_1)d\theta_1$,

$$\begin{aligned}
&\int_{\mathbb{S}_n} \left| \vec{b}\cdot\vec{\lambda_s}\right|\ d\vec{\lambda_s}\\
&= \int_0^{2\pi} d\theta_n \int_0^{\pi} sin(\theta_{n-1})d\theta_{n-1} \int_0^{\pi} sin^2(\theta_{n-2})d\theta_{n-2}... \int_0^{\pi} sin^{n-2}(\theta_2)d\theta_2 \int_0^{\pi} sin^{n-1}(\theta_1)\left|cos(\theta_1)\right|\ d\theta_1\\
&= \mathcal{S}_{n-1}\ 2\int_0^{\pi/2} sin^{n-1}(\theta_1)cos(\theta_1)\ d\theta_1 = 2\ \mathcal{S}_{n-1}[sin^n(\theta_1)/n]_0^{\pi/2}\\
&= \frac{2}{n}\ \mathcal{S}_{n-1}.
\end{aligned}$$

□

We will write $R_n = \frac{2}{n}\mathcal{S}_{n-1}$ to simplify notation.

Theorem 5 (Generalized sampling theorem). *Let $\vec{a}$ and $\vec{b} \in \mathbb{S}_n$ be Alice's and Bob's inputs. If Alice and Bob share a random variable $\vec{\lambda_s} \in \mathbb{S}_n$ distributed according to a biased distribution with probability density*

$$\rho(\vec{\lambda_s}|\vec{a}\vec{b}) = \rho_{\vec{a}}(\vec{\lambda_s}) = \frac{\left|\ \vec{a}\cdot\vec{\lambda_s}\right|}{R_n},$$

then they can simulate the joint correlations

$$E(AB|\vec{a}\vec{b}) = \vec{a}\cdot\vec{b},$$

with marginal expectations

$$E(A|\vec{a}\vec{b}) = E(B|\vec{a}\vec{b}) = 0.$$

This says that in this model, Alice and Bob can simulate these correlations without any further resource, that is, simulating the bipartite two output joint quantum correlations reduces to distributed sampling from the distribution $\rho_{\vec{a}}$.

Proof. Consider the protocol where Alice and Bob set their respective outputs to $A(\vec{a}\vec{\lambda_s}) = \text{sgn}(\vec{a}\cdot\vec{\lambda_s})$ and $B(\vec{b}\vec{\lambda_s}) = \text{sgn}(\vec{b}\cdot\vec{\lambda_s})$, where $\text{sgn}(x) = 1$ for $x \geq 0$ and $\text{sgn}(x) = -1$ for $x < 0$ ($x \in \mathbb{R}$). Then the joint expectation $E(AB|\vec{a}\vec{b})$ is given by

$$\begin{aligned}
E(AB|\vec{a}\vec{b}) &= \int_{\mathbb{S}_n} \rho_{\vec{a}}(\vec{\lambda_s})\ A(\vec{a},\vec{\lambda_s})B(\vec{b},\vec{\lambda_s})\ d\vec{\lambda_s}\\
&= \frac{1}{R_n}\int_{\mathbb{S}_n} \left|\vec{a}\cdot\vec{\lambda_s}\right|\ \text{sgn}(\vec{a}\cdot\vec{\lambda_s})\ \text{sgn}(\vec{b}\cdot\vec{\lambda_s})\ d\vec{\lambda_s}\\
&= \frac{1}{R_n}\int_{\mathbb{S}_n} (\vec{a}\cdot\vec{\lambda_s})\ \text{sgn}(\vec{b}\cdot\vec{\lambda_s})\ d\vec{\lambda_s}\\
&= \frac{1}{R_n}\vec{a}\cdot(\int_{\mathbb{S}_n} \vec{\lambda_s}\ \text{sgn}(\vec{b}\cdot\vec{\lambda_s})\ d\vec{\lambda_s}).
\end{aligned}$$

Observe that the final integral is invariant by rotation around $\vec{b}$, so it must be the case that

$$\int_{\mathbb{S}_n} \vec{\lambda_s}\ \text{sgn}(\vec{b}\cdot\vec{\lambda_s})\ d\vec{\lambda_s} = c\ \vec{b}, \tag{1}$$

with c a real constant.
Multiplying Equation 1 by $\vec{b}$ on either side to compute the constant, we obtain

$$\int_{\mathbb{S}_n} \vec{b}\cdot\vec{\lambda_s}\ \text{sgn}(\vec{b}\cdot\vec{\lambda_s})\ d\vec{\lambda_s} = c\ (\vec{b}\cdot\vec{b}) = c.$$

By Lemma 4, $c = R_n$, therefore,

$$E(AB|\vec{a}\vec{b}) = \frac{R_n}{R_n}\ \vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{b}.$$

Finally, we compute the marginal distributions. It is easy to see that

$$\begin{aligned} E(A|\vec{a}\vec{b}) &= \int_{\mathbb{S}_n} \rho_{\vec{a}}(\vec{\lambda_s})\ A(\vec{a},\vec{\lambda_s})\ d\vec{\lambda_s} \\ &= \frac{1}{R_n}\int_{\mathbb{S}_n} \left|\vec{a}\cdot\vec{\lambda_s}\right|\ \mathrm{sgn}(\vec{a}\cdot\vec{\lambda_s})\ d\vec{\lambda_s} \\ &= \frac{1}{R_n}\int_{\mathbb{S}_n} (\vec{a}\cdot\vec{\lambda_s})\ d\vec{\lambda_s} = 0. \end{aligned}$$

For the second marginal,

$$E(B|\vec{a}\vec{b}) = \int_{\mathbb{S}_n} \rho_{\vec{a}}(\vec{\lambda_s})\ B(\vec{b},\vec{\lambda_s})\ d\vec{\lambda_s} = \frac{1}{R_n}\int_{\mathbb{S}_n} \left|\vec{a}\cdot\vec{\lambda_s}\right|\ \mathrm{sgn}(\vec{b}\cdot\vec{\lambda_s})\ d\vec{\lambda_s}.$$

With $\mathbb{S}_+$, $\mathbb{S}_-$ the half-spheres with respect to $\vec{a}$, and with $\vec{\lambda_+} \in \mathbb{S}_+$ and $\vec{\lambda_-} \in \mathbb{S}_-$,

$$E(B|\vec{a}\vec{b}) = \frac{1}{R_n}\int_{\mathbb{S}_+} \vec{a}\cdot\vec{\lambda_+}\ \mathrm{sgn}(\vec{b}\cdot\vec{\lambda_+})\ d\vec{\lambda_+} - \frac{1}{R_n}\int_{\mathbb{S}_-} \vec{a}\cdot\vec{\lambda_-}\ \mathrm{sgn}(\vec{b}\cdot\vec{\lambda_-})\ d\vec{\lambda_-}.$$

We make a variable substitution $\vec{\lambda_-} = -\vec{\lambda_+}$ and we obtain

$$E(B|\vec{a}\vec{b}) = \frac{1}{R_n}\int_{\mathbb{S}_+} \vec{a}\cdot\vec{\lambda_+}\ \mathrm{sgn}(\vec{b}\cdot\vec{\lambda_+})\ d\vec{\lambda_+} - \frac{1}{R_n}\int_{\mathbb{S}_+} \vec{a}\cdot\vec{\lambda_+}\ \mathrm{sgn}(\vec{b}\cdot\vec{\lambda_+})\ d\vec{\lambda_+} = 0.$$

□

Note that we obtain $E(A|\vec{a}\vec{b}) = E(B|\vec{a}\vec{b}) = 0$. In the case of measurements on maximally entangled states, it is also the case that the marginal expectations are zero. This is because the reduced states of Alice and Bob are maximally mixed and, for traceless binary observables, the marginal distributions are uniform, so that $E(A|\vec{a}\vec{b}) = E(B|\vec{a}\vec{b}) = 0$. However, for arbitrary states, our method will reproduce the joint quantum correlations $E(AB|\vec{a}\vec{b})$, but not necessarily the marginal distributions.

B. Sampling the biased distribution: the rejection method

It now remains to show how Alice and Bob can obtain a shared sample $\vec{\lambda_s} \in \mathbb{S}_n$ distributed according to the biased distribution $\rho_{\vec{a}}(\vec{\lambda_s}) = \left|\ \vec{a}\cdot\vec{\lambda_s}\right|/R_n$, with help of shared $\vec{\lambda}$ uniformly distributed on $\mathbb{S}_n$, and with the additional help of communication. Using the same idea as [9, 14, 15], Alice uses the rejection method to perform the sampling.

Theorem 6. *There is a local hidden variable protocol that simulates the joint correlations of TBO measurements on a bipartite pair of d-dimensional states using* $\log(d) + O(1)$ *bits of communication on average.*

As noted above, for arbitrary states, we reproduce the joint correlations $E(AB|\vec{a}\vec{b})$, and in the special case of maximally entangled states, we reproduce the full joint distribution exactly.

Proof. By Theorem 5, it suffices to give a protocol to sample the distribution $\rho_{\vec{a}}$ on $\mathbb{S}_n$ for $n = 2d^2 - 1$ in a distributed fashion. We show that this can be achieved with $\log(d)$ communication on average. We obtain a sample by applying the rejection method [16], using unbiased (uniform) shared random variables.

Let $U(\vec{\lambda})$ be the uniform probability density function on $\mathbb{S}_n$, that is, $U(\vec{\lambda}) = 1/\mathcal{S}_n$. The protocol is as follows.

1. Alice obtains a uniform sample $\vec{\lambda} \sim U$

2. She computes $|\vec{a}\cdot\vec{\lambda}|$, and accepts $\vec{\lambda}$ with the corresponding probability. If she accepts, she sends Bob the iteration at which this occurred.

3. If she rejects, she starts over with a new sample $\vec{\lambda}$.

We compute the probability that Alice accepts at a given iteration (let us call this event "ok"), on average over the choice of $\vec{\lambda}$.

$$p(\text{ok}) = \int_{\mathbb{S}_n} p(\text{ok}|\vec{\lambda})\rho(\vec{\lambda})\ d\vec{\lambda}.$$

Since $\vec{\lambda}$ is uniformly distributed on $\mathbb{S}_n$, we have $\rho(\vec{\lambda}) = U(\vec{\lambda}) = 1/\mathcal{S}_n$. Moreover, we accept a given $\vec{\lambda}$ with probability $p(\text{ok}|\vec{\lambda}) = |\vec{a}\cdot\vec{\lambda}|$, so that

$$p(\text{ok}) \ = \ \frac{1}{\mathcal{S}_n}\int_{\mathbb{S}_n} |\vec{a}\cdot\vec{\lambda}|\ d\vec{\lambda} = \frac{\mathrm{R}_n}{\mathcal{S}_n}, \tag{2}$$

where we have used Lemma 4.

We may now compute the distribution of accepted $\vec{\lambda}$'s as follows

$$\begin{aligned}\rho(\vec{\lambda}|\text{ok}) \ &= \ \frac{\rho(\vec{\lambda})\ p(\text{ok}|\vec{\lambda})}{p(\text{ok})} \\ &= \ \frac{|\vec{a}\cdot\vec{\lambda}|}{\mathrm{R}_n},\end{aligned}$$

which corresponds to $\rho_{\vec{a}}$ as required. This proves that the protocol achieves its goal. It remains to show that it requires at most $O(\log(d))$ bits of communication on average.

The message sent in the protocol is the iteration i at which Alice accepts the current uniform sample. The distribution of the messages behaves according to a Poisson distribution P_p, that is, i is sent with probability

$$P_p(i) = (1-p)^{i-1}p,$$

where $p = p(\text{ok})$ in our case. To compute the number of bits sent on average, it suffices to compute the entropy of this distribution, which we do below. From Equation 2 and Lemma 4, we have

$$p(\text{ok}) \ = \ \frac{2\ \mathcal{S}_{n-1}}{n\ \mathcal{S}_n}.$$

The surface area $\mathcal{S}_n$ of the hyper-sphere $\mathbb{S}_n$ is given by

$$\mathcal{S}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})},$$

where Γ is the well known gamma function, defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

The acceptance probability is

$$p(\text{ok}) = \frac{2}{n\sqrt{\pi}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}.$$

Using the fact that [17, Ex. 9.44]

$$\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} = \sqrt{\frac{n}{2}} + O(\frac{1}{\sqrt{n}})$$

and in particular for any $n \geq 1$,

$$\frac{1}{2}\sqrt{\frac{n}{2}} \leq \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \leq \sqrt{\frac{n}{2}}.$$

Therefore

$$\sqrt{\frac{1}{2\pi n}} \leq p(\mathrm{ok}) \leq \sqrt{\frac{2}{\pi n}}.$$

The entropy of the messages is given by

$$\begin{aligned} H(P_p) &= \sum_i P_p(i) \ \log\left(\frac{1}{P_p(i)}\right) \\ &= \log\left(\frac{1}{p}\right) + \frac{1-p}{p}\log\left(\frac{1}{1-p}\right). \end{aligned}$$

So in our case, we get

$$\begin{aligned} H(P_p) &\leq \log(\sqrt{2\pi n}) + \frac{1-\sqrt{\frac{1}{2\pi n}}}{\sqrt{\frac{1}{2\pi n}}}\log\left(\frac{1}{1-\sqrt{\frac{2}{\pi n}}}\right) \\ &\leq \frac{1}{2}\log(n) + O(1). \end{aligned}$$

So, with $n = 2d^2 - 1$, $\log(d) + O(1)$ bits of communication are sufficient on average to simulate the joint quantum correlations of an arbitrary bipartite qudit pair, measured according to a TBO. □

IV. DISCUSSION AND CONCLUSION

We have shown that in the general case of bipartite qudit pairs, we can apply the sampling approach of [9] to simulate the joint quantum correlations, using $\log(d)$ bits of communication on average.

There are very few results in the literature concerning settings that are more general than projective measurements on maximally entangled qubit pairs. In the case of qudits, Bacon and Toner have shown how to simulate joint quantum correlations arising from TBO measurements with constant communication on average, but the correlations are simulated approximately [12], whereas here we simulate the correlations exactly. In [10], Massar *et al* gave a protocol that simulates the correlations of any local measurement on an arbitrary bipartite state exactly, but within a different model, which uses communication only and no local hidden variables.

In [9], we considered two ressources other than communication: post-selection, and nonlocal boxes. In the qubit setting, instead of iterating the rejection method until a suitable sample was selected, it could be proven that if the first sample failed the selection, then the second sample could be used. In this case, one bit of communication sufficed for Alice to communicate to Bob which sample to use, and the method could also be adapted to obtain a protocol that makes a single use of a nonlocal box.

It turns out that this so-called "choice method" does not extend directly to dimensions other than two. Therefore, we do not obtain a worst case analysis in this more general setting, nor do we get a protocol using nonlocal boxes.

On the other hand, our analysis immediately applies to protocols using post-selection, that is, where the protocol is allowed to abort with some probability. Here, the protocol succeeds with probability at least $\sqrt{\frac{1}{2\pi n}}$, where $n = 2d^2-1$, that is, $O(\frac{1}{d})$.

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